$SLDL^T$: a stochastic approach of the LDL^T decomposition to speed up Monte Carlo based simulations.

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1. Introduction

Nowadays, it is a common practice in the industry to use the finite element method in order to predict the structural behavior. Current tools for modelling physical systems are nowadays widespread and at a high level. The consequence is that the resulting models are becoming detailed and rather accurate, and this means that the complexity can be relatively high. When dealing with uncertainties, this complexity implies high computation cost especially using Monte Carlo simulations. In this context, model reduction is a topic which receives growing attention. Goals of model reduction are multiple: firstly, you have an internally complex system and you want to reduce it's complexity, preserving input-output behavior. Secondly, you are interested in systems and control theory or thirdly you want to make your simulations faster.

In the context of stochastic mechanic randomness is often simulated through Monte Carlo simulations. Even if Monte Carlo methods simulations have strong advantages [1, 2], they usually become extremely computationally extensive

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when full models are under consideration. That is the reason why alternatives have been developed such has the Stochastic Finite Element Method (SFEM) [3]. This method aims at representing the complete response probability density function (PDF) and is based on the discretization of the input random fields and the expansion of the response onto a particular basis of the probability space called the polynomial chaos [3]. This is equivalent to build a metamodel.

In this paper we focus on model based approaches instead of metamodelling techniques. Model based approaches deal with the mechanical model and its resolution. We introduce an original use of the LDL^T decomposition to accelerate the inversion of the stiffness matrix. This technique is developed within the context of linear elastic mechanic with material properties modeled by a stochastic field. Mathematical aspects are first described. Then, the method is applied onto an industrial example from the rail industry.

2. The Cholesky decomposition

2.1. Introduction

The Cholesky decomposition is a decomposition of a symmetric, positive-definite matrix A into the product of a lower triangular matrix and its conjugate transpose [4, 5]:

$$A = LL^T \tag{1}$$

where L is a lower triangular matrix with strictly positive diagonal entries. The Cholesky decomposition is unique: given a symmetric positive-definite matrix A, there is only one lower triangular matrix L with strictly positive diagonal entries such that $A = LL^T$ (in the case of a positive semi-definite matrix A, the Cholesky decomposition is not unique). An alternative form is the factorization [6, 7]:

$$A = LDL^T$$

This form eliminates the need to take square roots:

$$
\begin{aligned}
L_{i,j} &= \frac{1}{D_j}\left(A_{i,j} - \sum_{k=1}^{j-1} L_{i,k} L_{j,k} D_k\right) \qquad for\, i > j \\
D_i &= A_{i,i} - \sum_{k=1}^{i-1} L_{i,k}^2 D_k
\end{aligned} \tag{2}
$$

Because A is positive definite, the elements of the diagonal matrix D are all positive. However this factorization can be used for any square, symmetrical matrix.

2.2. Application to Monte Carlo simulations

The Cholesky decomposition is commonly used in the Monte Carlo simulations for simulating systems with correlated variables or random fields [8]. The correlation matrix is first decomposed using Cholesky decomposition. Then, the lower triangular matrix L is applied to a vector $\underline{\xi}$ of uncorrelated variables. The product $L\,\underline{\xi}$ produces a sample vector with the covariance properties of the system being modeled. In this paper, the Young modulus is supposed to be modeled by a gaussian random field of mean μ_E, standard deviation σ_E and a variance-covariance matrix Σ_E (which is a symmetric and positive semi-definite matrix). The correlation kernel is supposed to be Gaussian too. Let $\underline{\xi}$ be a vector (of size n) whose components are independent standard normal random variables. L is the matrix obtained from the Cholesky decomposition of the variance-covariance matrix Σ_E. The random vector E follows a multivariate normal distribution and $E = \mu_E + L\underline{\xi}$. The vector μ_E in these conditions is the expected value of E and the matrix $\Sigma_E = L\,L^T$ is the covariance matrix of the components E_i. If Σ_E is non-singular, then the distribution may be described by the following PDF:

$$
f_E(e_1, \ldots, e_n) = \frac{1}{(2\pi)^{n/2} \det\left(\Sigma_E\right)^{1/2}} \exp\left(-\frac{1}{2}\left[e - \mu_E\right]^T \Sigma_E^{-1}\left[e - \mu_E\right]\right)
$$

2.3. Application to finite elements simulations

In finite elements analysis, the stiffness matrix is a banded semi-positive definite matrix. In this case, the LDL^T decomposition is very usefull to solve the system of linear equations. Let us considered a Young modulus modeled by a random field $E(\underline{x}, \theta)$. The loads f are supposed to be deterministic (however, assuming randoms load do not affect the method and is not considered here for clarity). Using the finite element formulation [9] in the context of linear elasticity, the following system must be solved:

$$K(E(\underline{x}, \theta))\, u(E(\underline{x}, \theta)) = f \tag{3}$$

The resolution of equation 3 is done using the LDL^T decomposition of the matrix $K(E(\underline{x}, \theta))$. By construction, the stiffness matrix K is symmetric, positive definite and band. It admits a LDL^T decomposition [7]:

$$K(E(\underline{x}, \theta)) = L(E(\underline{x}, \theta))\, D(E(\underline{x}, \theta))\, L(E(\underline{x}, \theta))^T \tag{4}$$

$L(E(\underline{x}, \theta))$ is a lower triangular matrix with unit diagonal entries and $D(E(\underline{x}, \theta))$ is a diagonal matrix which diagonal terms are noted d. Injecting 4 into 3, the displacement field $\underline{u}(E(\underline{x}, \theta))$ reads:

$$u(E(\underline{x}, \theta)) = L(E(\underline{x}, \theta))^{-T}\, D(E(\underline{x}, \theta))^{-1}\, L(E(\underline{x}, \theta))^{-1}\, f \tag{5}$$

Remark 1 For reading purpose, we write $E(\underline{x}, \theta)$ as E until the end of the document.

Remark 2 In the context of Monte Carlo simulations, the LDL^T decomposition has to be performed for each sample which is computationally intensive. If b is the half bandwidth of the stiffness matrix, n_{dof} the number of degrees of freedom, n_{elem} the number of elements of the finite element model and n_{samp} the number of samples; the cost of a classic Monte Carlo resolution is equal to the cost of a Cholesky decomposition ($n_{dof}b^2 + 2n_{dof}b$) plus forward and

backward substitution ($4n_{dof}b$) plus the cost of the inversion of the matrx D (n_{dof}). The total cost (for one sample) of the classic Monte Carlo approach is $n_{dof}b^2 + 6n_{dof}b$. However, in the context of stochastic computation, in order to take advantage of this decomposition, we proposed a modified version of the LDL^T decomposition: the $SLDL^T$ decomposition.

3. $SLDL^T$ decomposition

3.1. Hypothesis

The idea behind the $SLDL^T$ decomposition is to approximate equation 4 with the following formula:

$$K^{ap}(E) = L(E_0)D(E)L(E_0)^T = L_0D(E)L_0^T \tag{6}$$

$K^{ap}(E)$ is the LDL^T approximation of the stiffness matrix $K(E)$, E_0 corresponds to the mean value of the stochastic field $E(\underline{x}, \theta)$ ($E_0 = E(\underline{x}, \theta_0)$). The form in equation 6 is equivalent to make the assumption that the fluctuations of the matrix L are supposed to be negligible. This way, the randomness is put on the diagonal matrix D. The matrix L_0 is computed from the matrix $K_0 = K(E_0)$:

$$K_0 = L_0D(E_0)L_0^T \tag{7}$$

3.2. Optimization problem

From equation 6, for each sample of the random field, the objective is to calculate the best matrix $D(E)$ which approximates the stiffness matrix $K(E)$. This is equivalent to solve an optimization problem: minimize the difference between $K(E)$ and $K^{ap}(E)$. To help computation, we use the following lemma:

Lemma 1 The matrix $L_0D(E)L_0^T$ can be written as a combination of the terms d_i of the diagonal matrix D as:

$$L_0 D(E) L_0^T = \sum_{i=1}^{n_{dof}} d_i(E) \Lambda^i \tag{8}$$

where n_{dof} is the number of degrees of freedom and $\Lambda^i = (L_0)_{.i} (L_0)_{.i}^T$ (product of the *ith* column of L_0 with the *ith* row of L_0^T).

3.2.1. Matrix preconditioning

In order to reduce the condition number of a matrix, the use of a preconditioner can be very usefull. To preserve matrix symmetry, a left-right preconditioning technique is used. We use the Jacobi preconditioner which consist of the diagonal of the matrix [10]:

$$m_{ij} = \begin{cases} \sqrt{a_{ij}} & \text{if} \quad i = j \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

For instance, if a matrix M approximates the coefficients matrix A in some way, the system:

$$M^{-1} A M^{-1} (Mx) = M^{-1} b \tag{10}$$

has the same solution as the original system $Ax = b$, but the spectral properties of its coefficient matrix $M^{-1}AM^{-1}$ may be more favorable.

3.2.2. Optimality system

In the context of linear mechanic, the matrix $K(E)$ is a linear function of E. This way, the matrix $D(E)$ (see equation 6) is defined as a linear form in E and the optimization criterion must be a quadratic one. As the difference between $K(E)$ and $K^{ap}(E)$ must be minimized, the cost function $J_E(d)$ writes:

$$J_E(d) = \frac{1}{2} \left(M(E)^{-1} K(E) M(E)^{-1}, L_0 D(E) L_0^T \right)_F \tag{11}$$

$(,)_F$ is the Frobenius norm: $(,)_F = trace\left\{AA^T\right\}$. The preconditioning matrix $M(E)$ is assumed to be constant and equal to M_0:

$$M(E) = M\left(E_0\right) = M_0 = diag(K_0)^{1/2} \tag{12}$$

Injecting equation 12 into equation 11 and using equation 8, the cost function $J_E(d)$ rewrites:

$$J_E(d) = \frac{1}{2}\left(M_0^{-1}K(E)M_0^{-1}, \sum_{i=1}^{n_{dof}} d_i(E)\Lambda^i\right)_F \tag{13}$$

Assuming $K_{cond}(E) = M_0^{-1}K(E)M_0^{-1}$ equation 13 is simplified:

$$J_E(d) = \frac{1}{2}\left(K_{cond}(E) - \sum_{i=1}^{n_{dof}} d_i(E)\Lambda^i\right)_F \tag{14}$$

3.2.3. Optimality condition

The objective function $J_E(d)$ (equation 14) is strictly convex and differentiable in d so a necessary and sufficient optimality condition writes:

$$J'_E(d)\delta d = 0 \quad \forall \delta d \tag{15}$$

This equivalent to solve the system:

$$trace\left\{-\left(K_{cond}(E) - \sum_{j=1}^{n_{dof}} d_j(E)\Lambda^j\right)\Lambda^i\right\} = 0 \quad \forall i = \{1, \ldots, n_{dof}\} \tag{16}$$

Rewriting and rearranging equation 16, the solution of the optimization problem (equation 11) is:

$$Ad(E) = b(E) \tag{17}$$

With:

$$\begin{aligned} A_{i,j} &= trace\left\{\Lambda^j\Lambda^i\right\} = \left[\left(L_0\right)_{.j}^T\left(L_0\right)_{.i}\right]^2 \\ b_i(E) &= trace\left\{K_{cond}(E)\Lambda^i\right\} = \left(L_0\right)_{.i}^T K_{cond}(E)\left(L_0\right)_{.i} \end{aligned} \tag{18}$$

The matrix A is a band matrix with the same bandwidth as K (the LDL^T decomposition preserves the band structure), is independent of the random field $E(\underline{x}, \theta)$ and has the same size as K.

3.2.4. Algorithm improvements

Equation 18 can be rewritten to speed up the computation of the matrix $D(E)$ considering the n_{elem} elementary stiffness matrices k^e and assuming the uniformity of the Young modulus over the finite element e:

$$K_{cond}(E) = M_0 \left(\mathbb{A}_{e=1}^{n_{elem}} k^e\right) M_0 = \sum_{e=1}^{n_{elem}} E(e) K_{cond}^e \tag{19}$$

$E(e)$ is the Young modulus associated to the *eth* element. K_{cond}^e is a $n_{dof} \times n_{dof}$ in which the elementary stiffness matrix k^e is positioned. k^e is computed for a unit Young modulus. Following this assumption, equation 18 rewrites ($\forall i = \{1, \ldots, n_{dof}\}$):

$$b_i(E) = (L_0)_{.i}^T \left(\sum_{e=1}^{n_{elem}} E(e) K_{cond}^e \right) (L_0)_{.i} = \sum_{e=1}^{n_{elem}} E(e) b_i^e \tag{20}$$

which finally leads to:

$$b(E) = \sum_{e=1}^{n_{elem}} E(e) b^e \tag{21}$$

Injecting equation 21 into equation 17, the solution of the system writes:

$$d(E) = A^{-1} b(E) = \sum_{e=1}^{n_{elem}} E(e) d^e \tag{22}$$

with:

$$d^e = A^{-1} b^e$$

3.3. Displacements computation

Equation 22 gives the best coefficients for the diagonal matrix D. Assuming a deterministic load and injecting equation 6 into 3, the displacement field $u(E)$ of the structure is equal to:

$$u(E) = L_0^{-T} D^{-1}(E) L_0^{-1} f \tag{23}$$

In this expression of the displacement field, only the matrix $D^{-1}(E)$ has to be updated. This way, the number of flops is equal to backward and forward substitution plus the inversion of $D(E)$ (total of $4n_{dof}b$ flops).

3.4. Error control

The goal of error estimators is to estimate the error made by the approximation. In the case of Monte Carlo LDL^T simulation, two *a priori* estimators have been developed: one for the approximation of the stiffness matrix and one for the displacement field. It defines the error as the difference between the exact solution and the one calculated by the LDL^T decomposition.

3.4.1. Error estimator for the stiffness matrix

The error estimator for the stiffness matrix computation is deduced from the resolution of equation 14 for the optimal value d^{opt}:

$$\begin{aligned} J_E(d^{opt}) &= \frac{1}{2}\left(K_{cond}(E) - \sum_{i=1}^{n_{dof}} d_i^{opt}(E)\Lambda^i, K_{cond}(E) - \sum_{j=1}^{n_{dof}} d_j^{opt}(E)\Lambda^j \right)_F \\ &= \frac{1}{2}\left(K_{cond}(E) - \sum_{i=1}^{n_{dof}} d_i^{opt}(E)\Lambda^i, K_{cond}(E) \right)_F \\ &= \frac{1}{2}\left(K_{cond}(E), K_{cond}(E) \right)_F - \frac{1}{2}\sum_{i=1}^{n_{dof}} d_i^{opt} b_i(E) \end{aligned} \tag{24}$$

This equation can be rewritted injecting equation 17:

$$\begin{aligned} J_E(d^{opt}) &= \frac{1}{2}\left(K_{cond}(E), K_{cond}(E) \right)_F - \frac{1}{2}\left(A^{-1} b(E) \right)^T b(E) \\ &= \frac{1}{2}\left(K_{cond}(E), K_{cond}(E) \right)_F - \frac{1}{2} b^T(E) d(E) \end{aligned} \tag{25}$$

Finally, equations 19 and 25 lead to:

$$J_E(d^{opt}) = \frac{1}{2} E^T \, Q \, E \tag{26}$$

Nevertheless, it is possible to simplify the computation of matrix Q from equation 25:

$$Q = Q^1 - Q^2$$

with:

$$\begin{aligned} Q^1_{ee'} &= \frac{1}{2} \left(K^e_{cond}, K^{e'}_{cond} \right)_F \\ Q^2_{ee'} &= \frac{1}{2} \left(b^e \right)^T d^{e'} \end{aligned} \tag{27}$$

The Q matrix is independent of the realizations of the stochastic field. A slight modification of equation 26 allows to simplify the expression of the error estimator:

$$\begin{aligned} J_E(d^{opt}) &= \frac{1}{2} \sum_{e=1}^{n_{elem}} \sum_{e'=1}^{n_{elem}} E_e \, E_{e'} Q_{ee'} \\ &= \frac{1}{2} \sum_{e=1}^{n_{elem}} \sum_{e'=1}^{n_{elem}} \left(E \, E^T \right)_{ee'} Q_{ee'} \\ &= \frac{1}{2} \left(E \, E^T, Q \right)_F \end{aligned} \tag{28}$$

From equation 28, the mean of the estimator can be directly computed from the knowing of the covariance matrix $Cov(E)$ (wihch is one data). By definition:

$$Cov(E) = \mathbb{E}_\theta \left\{ \left(E - E_0 \right) \left(E - E_0 \right)^T \right\} \tag{29}$$

which implies:

$$\mathbb{E}_\theta \left(E \, E^T \right) = Cov(E) + E_0 \, E_0^T \tag{30}$$

Consequently, the mean error rewrites as (from equation 28):

$$\begin{aligned}\mathbb{E}_\theta\left(J_E(d^{opt})\right) &= \frac{1}{2}\left(\mathbb{E}_\theta\left(E\,E^T\right),Q\right)_F \\ &= \frac{1}{2}\left(Cov(E),Q\right)_F + \frac{1}{2}\left(E_0\,E_0^T,Q\right)_F\end{aligned} \tag{31}$$

Besides, the unicity of the optimum of J implies $\left(E_0\,E_0^T,Q\right)_F = 0$. Finally, the solution is:

$$\mathbb{E}_\theta\left(J_E(d^{opt})\right) = \left(Cov(E),Q\right)_F \tag{32}$$

$Cov(E)$ is one input of the problem and the matrix Q is independent of E sampling. This way, $\mathbb{E}_\theta$ can be estimated *a priori*.

3.4.2. Error estimator for the displacement field

The objective is to estimate $\parallel u^{ex}(E) - u^{ap}(E) \parallel_{L^2}$ which is the L^2 norm of the difference between the exact value u^{ex} of u and its LDL^T approximation u^{ap}. For one sample of the Young modulus field $K^{ex}u^{ex} = f$ and $K^{ap}u^{ap} = f$. This way for every realization:

$$K^{ex}u^{ex} - K^{ap}u^{ap} = 0 \tag{33}$$

This equation can be rearranged:

$$\left[K^{ex} - K^{ap}\right]u^{ex} + K^{ap}\left(u^{ex} - u^{ap}\right) = 0 \tag{34}$$

and:

$$\begin{aligned}u^{ex} - u^{ap} &= \left(K^{ap}\right)^{-1}\left[K^{ex} - K^{ap}\right]u^{ex} \\ &= \left[I - \left(K^{ap}\right)^{-1}K^{ex}\right]u^{ex}\end{aligned} \tag{35}$$

Finally, from equation 35:

$$\frac{\| u^{ex} - u^{ap} \|_{L^2}}{\| u^{ex} \|_{L^2}} \leqslant \| (K^{ap})^{-1} [K^{ap} - K^{ex}] \|_{L^2} \tag{36}$$

where $\|A\|_{L^2} = \left[\rho\left(A^T A\right)\right]^{1/2}$ is the matrix norm related to the euclidian norm. Introducing the following lemma (lemma 2) it is possible to rewrite equation 36.

Lemma 2 If K is a symmetric positive definite matrix which admits a LDL^T decomposition then:

$$\|K\|_{L^2} \leqslant l\, d_{max} \tag{37}$$

with $d_{max} = \max_{i=1,\ldots,n_{dof}} d_i$ and $l = \rho\left(L\, L^T\right)$.

Lemma 2 demonstration

$$\begin{aligned} \|K\|_{L^2} &= \|LDL^T\|_{L^2} \\ &\leqslant \|L\|_{L^2}\, \|D\|_{L^2}\, \|L^T\|_{L^2} \\ &\leqslant \|L\|_{L^2}^2\, \|diag(d)\|_{L^2} \end{aligned} \tag{38}$$

The computation of $\| u^{ex} - u^{ap} \|_{L^2} \,/\, \| u^{ex} \|_{L^2}$ is equivalent to the computation of the relative error which is noted $errel_u(E)$. Using lemma 2, the relative error on u writes:

$$errel_u \leqslant \gamma \max_i (1/d_i)\, \|K^{ap} - K^{ex}\|_{L^2} \tag{39}$$

with $\gamma = \|L_0^{-1}\|^2$. From norms equivalence between the $\mathbb{L}^2$-norm and the Frobenius norm, injecting equation 28 into equation 39 the relative error on u is equal to:

$$errel_u \leqslant \gamma\ (1/d_{min})\ \left(E\, E^T, Q\right)_F^{1/2}$$

3.5. Statistical moments evaluation

One of the advantages of the proposed LDL^T decomposition is to allow the computation of the mean and the covariance matrix of the displacement field

without mechanical computations. The mean $\bar{u}$ and the covariance matrix C_u are directly deduced from equation 23 and coefficients $1/d_i(E)$.

3.5.1. The mean

$$\begin{aligned}
\bar{u} &= \int_\Omega u(E) dp\theta \\
&= \int_\Omega K^{-1}(E)\, f dp\theta \\
&= L_0^{-T} \int_\Omega D^{-1}(E)\, L_0^{-1}\, f\, dp\theta \\
&= L_0^{-T} \int_\Omega D^{-1}(E)\, L_0^{-1} dp\theta\, f \\
&= L_0^{-T} \left\{ \int_\Omega D^{-1}(E) dp\theta \right\} L_0^{-1}\, f \\
&= L_0^{-T}\, \bar{D}^{-1}\, L_0^{-1}\, f
\end{aligned} \tag{40}$$

with:

$$\bar{D}^{-1} = \begin{cases} \bar{D_{ii}}^{-1} = \int_\Omega \frac{1}{d_i(E)} dp\theta \\ \bar{D_{ij}}^{-1} = 0 \end{cases} \tag{41}$$

3.5.2. Covariance matrix

By definition:

$$\begin{aligned}
C_u &= \int_\Omega \left(u(E) - \bar{u}\right) \left(u(E) - \bar{u}\right)^T dp\theta \\
&= \int_\Omega u(E)\, u(E)^T dp\theta - \int_\Omega u(E)\, \bar{u}^T dp\theta - \int_\Omega \bar{u}\, u(E)^T dp\theta + \int_\Omega \bar{u}\, \bar{u}^T dp\theta \\
&= L_0^{-T} C_d L_0^{-1} - \int_\Omega u(E) dp\theta\, \bar{u}^T - \bar{u} \int_\Omega u(E)^T dp\theta + \bar{u}\, \bar{u}^T \int_\Omega dp\theta \\
&= L_0^{-T} C_d L_0^{-1} - \bar{u}\, \bar{u}^T
\end{aligned} \tag{42}$$

The matrix C_d is detailled bellow:

$$
\begin{aligned}
\int_\Omega u(E)\, u(E)^T dp\theta &= \int_\Omega K^{-1}(E)\, f\, f^T\, K^{-T}(E) dp\theta \\
&= \int_\Omega K^{-1}(E)\, C_f\, K^{-T}(E) dp\theta \\
&= \int_\Omega L_0^{-T}\, D^{-1}(E)\, L_0^{-1}\, C_f\, L_0^{-T}\, D^{-1}(E)\, L_0^{-1} dp\theta \\
&= L_0^{-T} \left\{ \int_\Omega D^{-1}(E)\, C_g\, D^{-1}(E) dp\theta \right\} L_0^{-1} \\
&= L_0^{-T}\, C_d\, L_0^{-1}
\end{aligned}
$$

with:

$$
\begin{aligned}
C_f &= f\, f^T \\
C_g &= L_0^{-1}\, C_f\, L_0^{-T} \\
(C_d)_{ij} &= \int_\Omega D_{ii}^{-1}\, (C_g)_{ij}\, D_{ii}^{-1} dp\theta \\
&= \left\{ \int_\Omega \frac{dp\theta}{d_i(E)\, d_j(E)} \right\} (C_g)_{ij}
\end{aligned}
\tag{43}
$$

3.6. Computational aspects

In the $SLDL^T$ approach, the cost of the Cholesky decomposition is replaced by the resolution of equation 22 which is $2n_{dof}^2$. Nevertheless, the cost of the error estimation has to be added (equation 28). From equation 27, the cost of the Q matrix computation is equal to $4n_{elem}n_{dof}b$. This matrix has to be computed once. Now, from equation 28 the cost of the estimation for one sample is deduced and equal to $2n_{elem}^2$. Finally the gain is equal to:

$$
Gain = \frac{n_{samp} n_{dof} b^2}{\left(2n_{dof}^2 + 2n_{elem}^2\right) n_{samp} + 4n_{elem} n_{dof} b} \tag{44}
$$

This formula is very usefull because you can predict *a priori* the performance of the method including the cost of model error estimation.

4. Application: Bogie support fixing

The proposed Monte Carlo LDL^T decomposition is applied onto an industrial example from the rail industry: a bogie support fixing (figure 1). The mesh (generated by ANSYS, see figure 2) is composed of $n_{elem} = 3408$ 10-Node Tetrahedral elements which leads to 7058 nodes and 21174 unconstrained degrees of freedom. Applying the boundaries conditions (figure 1) the size of the system is $n_{dof} = 20841$ degrees of freedom. The pattern of the stiffness matrix is plotted on figure 3. The maximal half bandwidth is $b_{max} = 2342$ and the mean is $b = 584$. The pressure is deterministic. The Young modulus is modeled by a Gaussian random field with Gaussian correlation model. The influence of the coefficient of variation is investigated ($C_v = \{0.01, 0.05\}$) as well as the performances of the method in terms of computational gain and error.

Les résultats pour $C_v = 0.01$ sont présentés Figures 4 à 8. La Figure 4 représente l'histogramme de l'erreur relative calculée par rapport aux tirages de Monte Carlo. L'histogramme montre que l'erreur relative maximale est de 5% mais aussi que globalement l'erreur est inférieure à 2% avec une moyenne à 1.15%. Pour l'erreur calculée par l'estimateur d'erreur (équation 39), la moyenne estimée est de 473 pour un maximum de 1431 (Figure 5. Cependant, si l'on regarde le tracé de l'erreur relative par rapport à l'erreur estimée (Figure 6), il n'y a pas de corrélation entre l'erreur relative et l'erreur estimée. Ceci montre que même s'il est exacte du point de vue mathématique, la majoration induite par l'estimateur ne permet pas de conclure directement sur la qualité du tirage. La Figure 7 montre l'évolution du gain et du temps de calcul en fonction du nombre de tirages rejetés. A partir de 5% d'erreur, le gain plafonne à 22 et le temps de calcul est alors égale au temps de calcul $SLDL^T$. Si l'on regarde la courbe de convergence de la moyenne de l'erreur relative en fonction du niveau d'erreur (Figure 8), à partir de 5% d'erreur la valeur de référence est atteinte. En conclusion, pour un C_v de 1%, l'erreur maximale sur le déplacement est de 5% pour un facteur gain de 22.

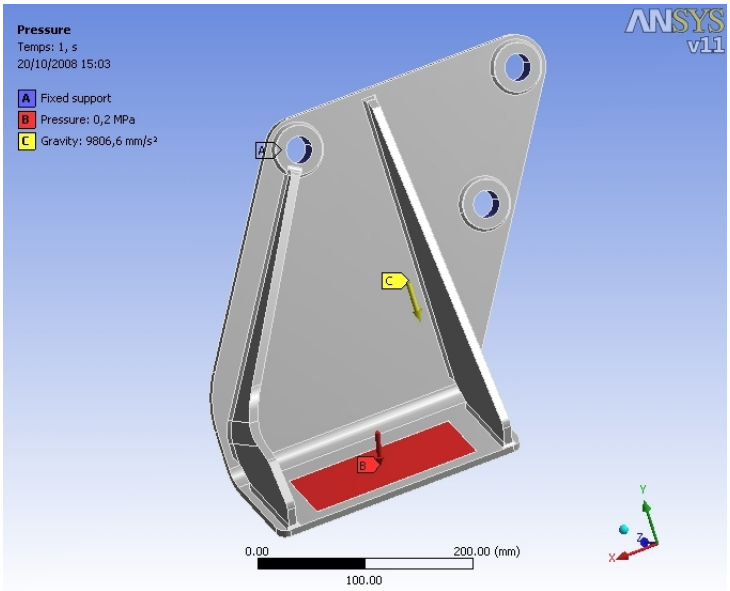

Figure 1: Bogie support fixing - Boundary conditions.

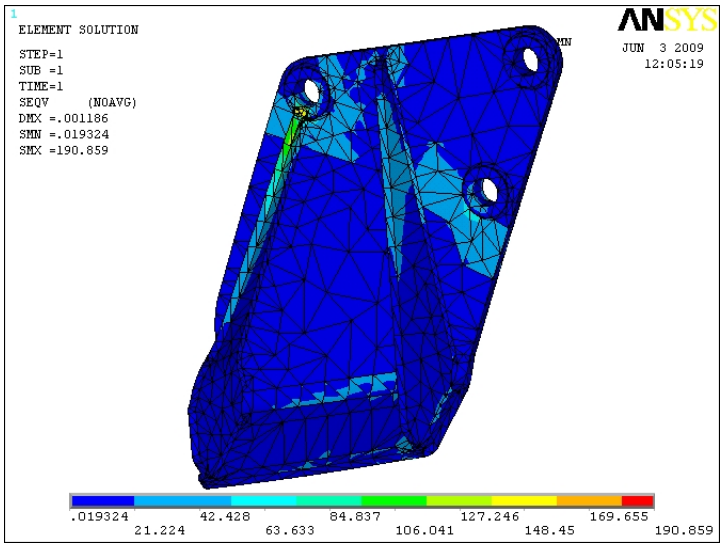

Figure 2: Bogie support fixing - Mesh and Stress analysis results.

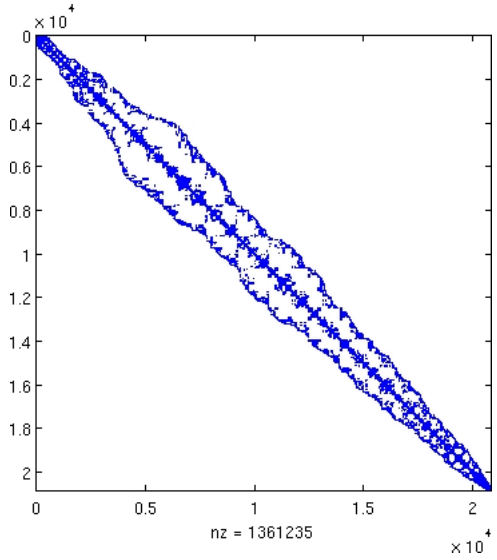

Figure 3: Bogie support fixing - Stiffness matrix shape.

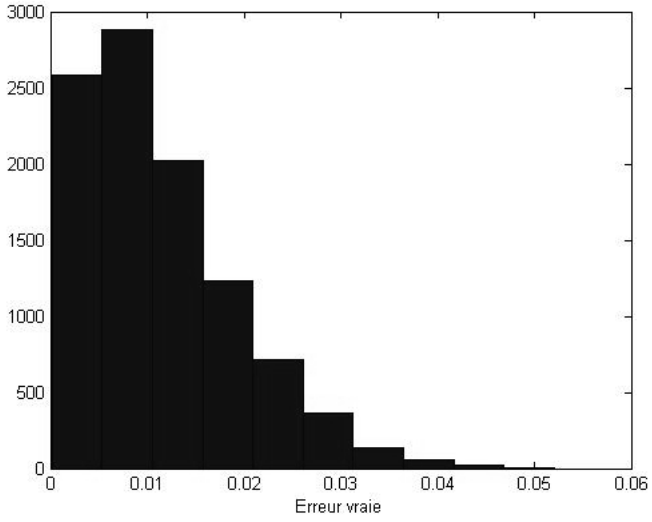

Figure 4: Bogie - Histogramme de l'erreur relative sur u calculée par rapport aux tirages Monte Carlo pour $C_v = 0.01$.

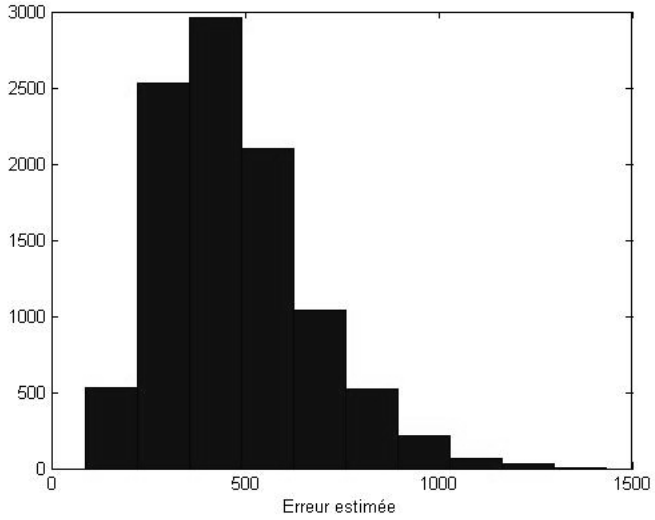

Figure 5: Bogie - Histogramme de l'erreur sur u calculée par l'estimateur pour $C_v = 0.01$.

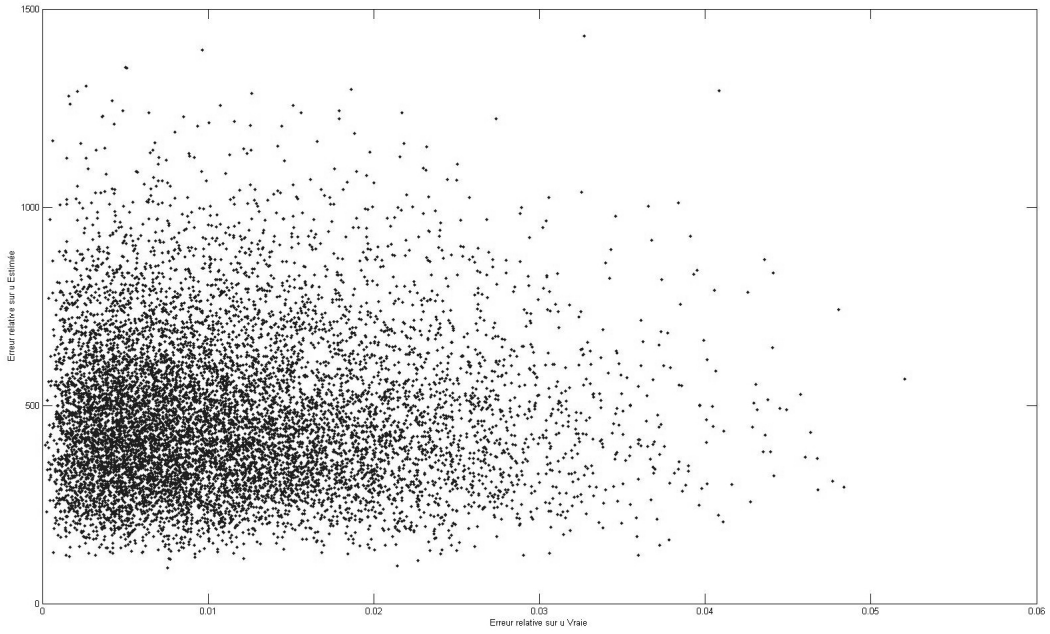

Figure 6: Bogie - Représentation de l'erreur relative par rapport à l'erreur estimée pour $C_v = 0.01$.

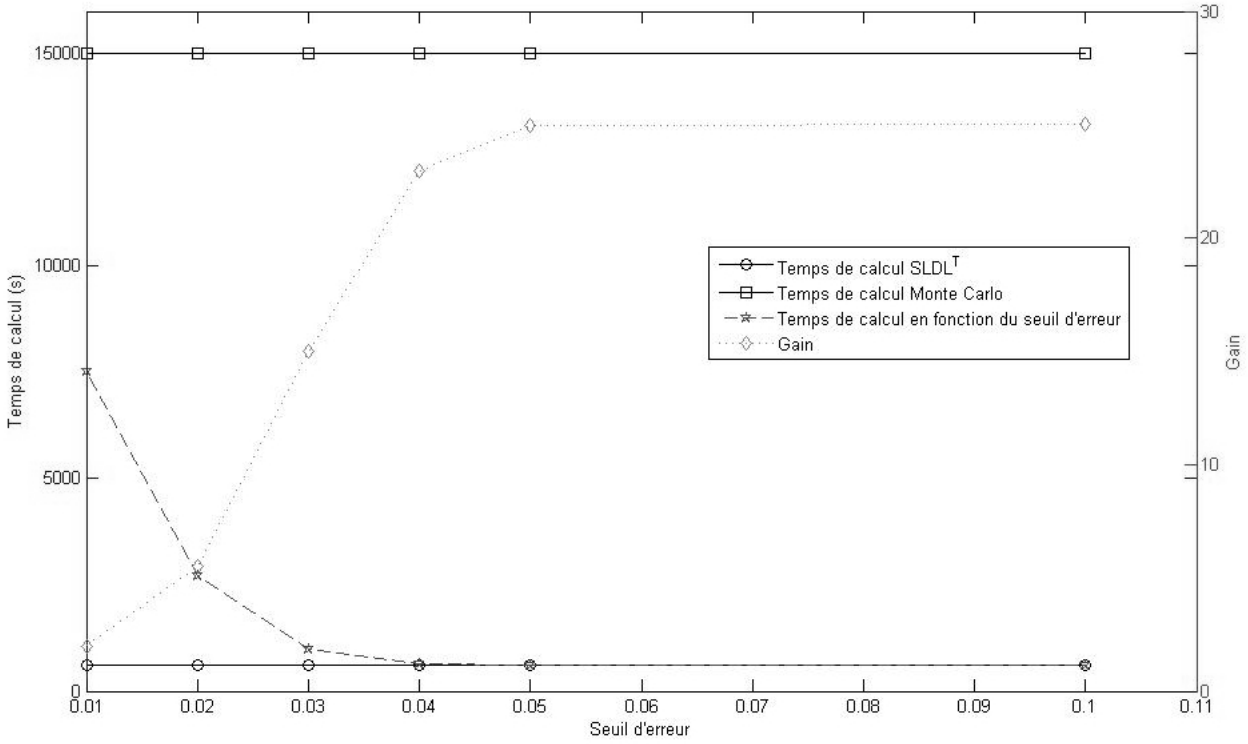

Figure 7: Bogie - Evolution du gain et du temps de calcul en fonction du seuil d'erreur pour $C_v = 0.01$.

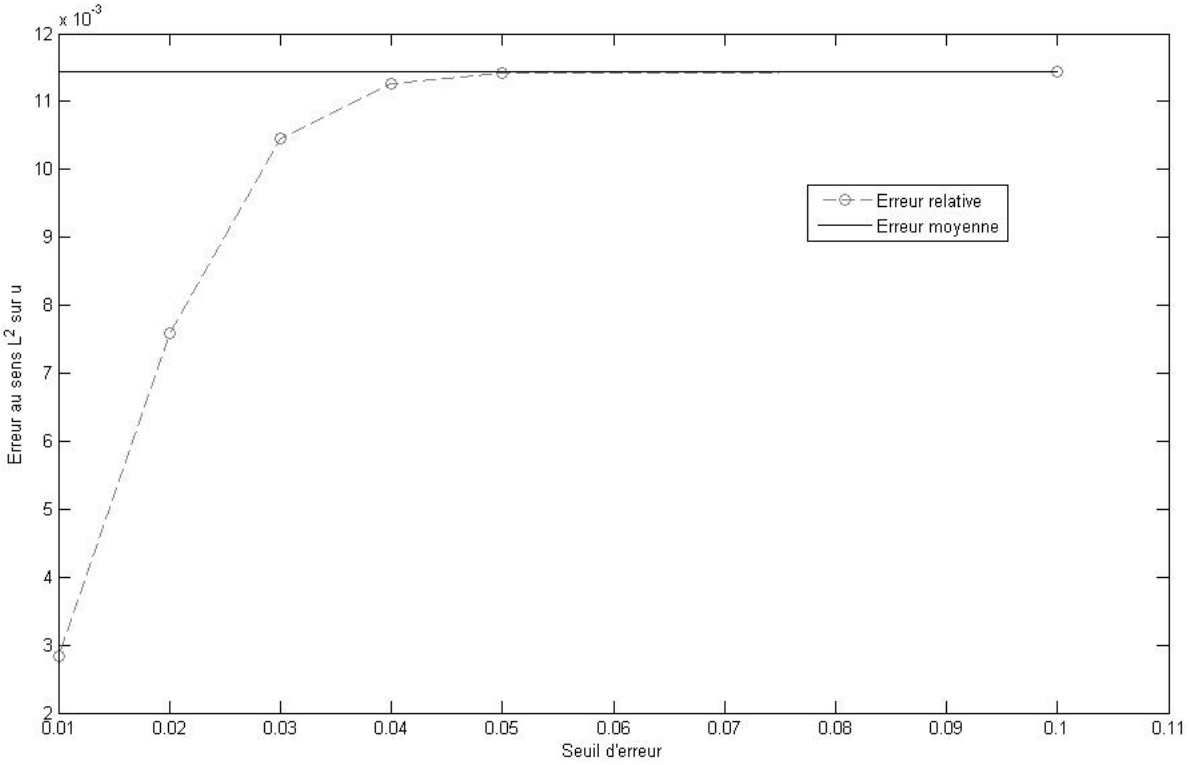

Figure 8: Bogie - Graphe de convergence en moyenne de l'erreur relative en fonction de l'erreur pour $C_v = 0.01$.

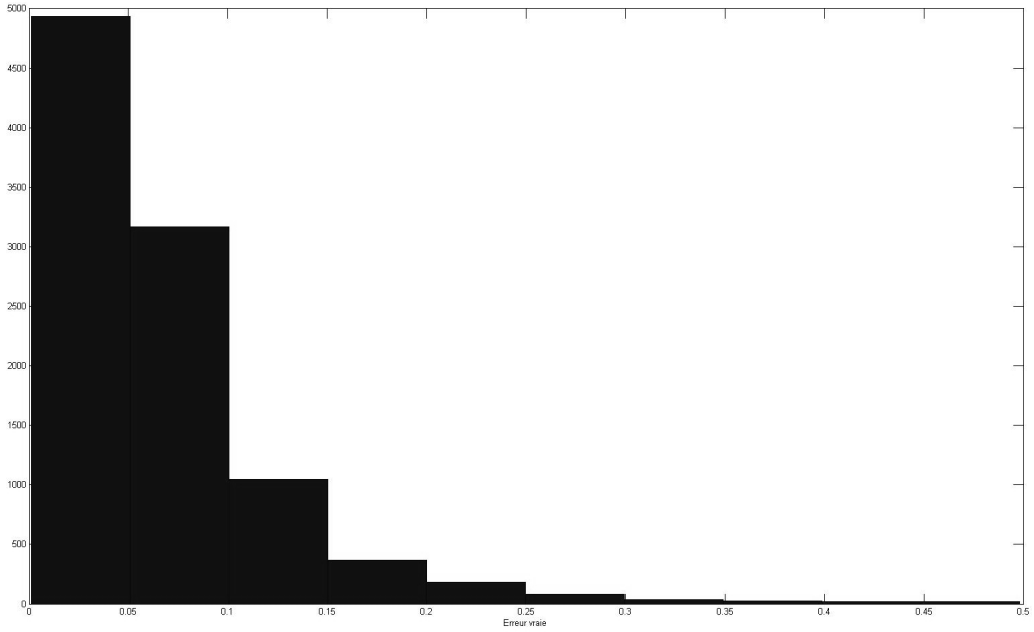

Figure 9: Bogie - Histogramme de l'erreur relative sur u calculée par rapport aux tirages Monte Carlo pour $C_v = 0.05$.

Figure 10: Bogie - Histogramme de l'erreur sur u calculée par l'estimateur pour $C_v = 0.05$.

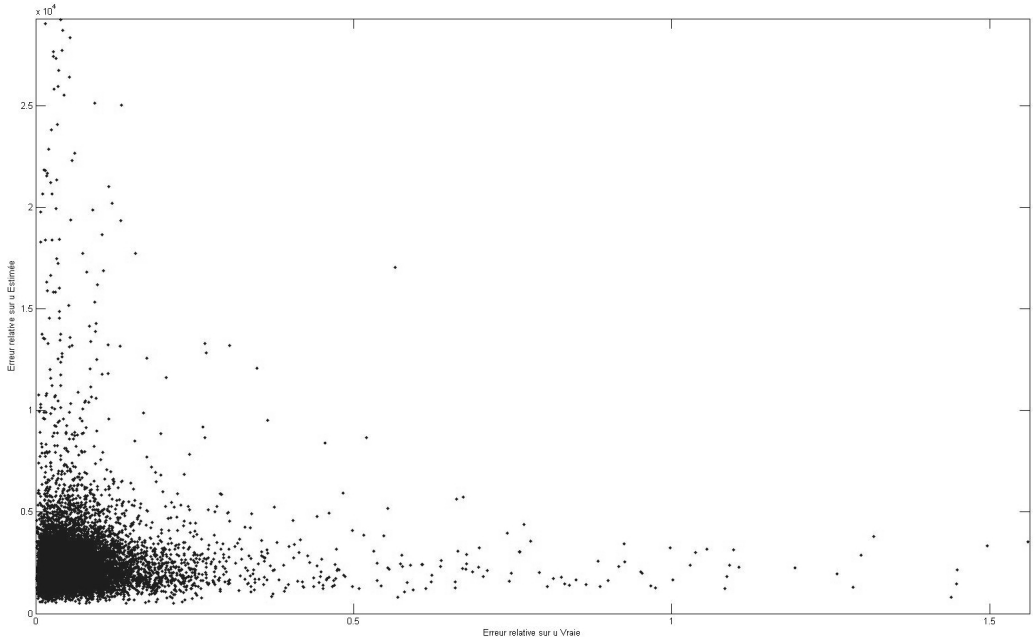

Figure 11: Bogie - Représentation de l'erreur relative par rapport à l'erreur estimée pour $C_v = 0.05$.

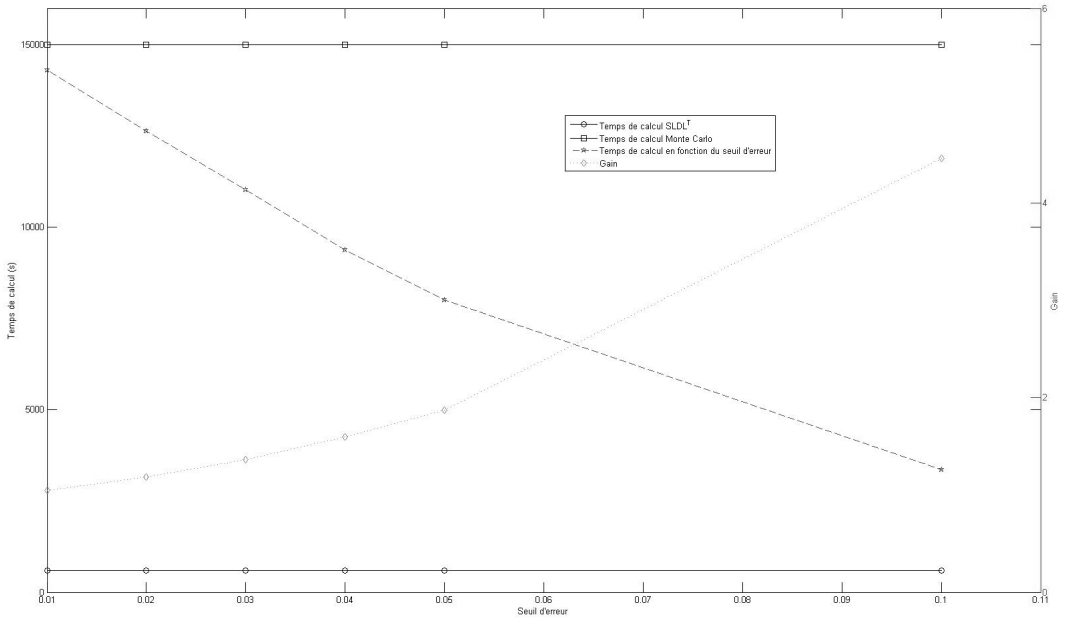

Figure 12: Bogie - Evolution du gain et du temps de calcul en fonction du seuil d'erreur pour $C_v = 0.05$.

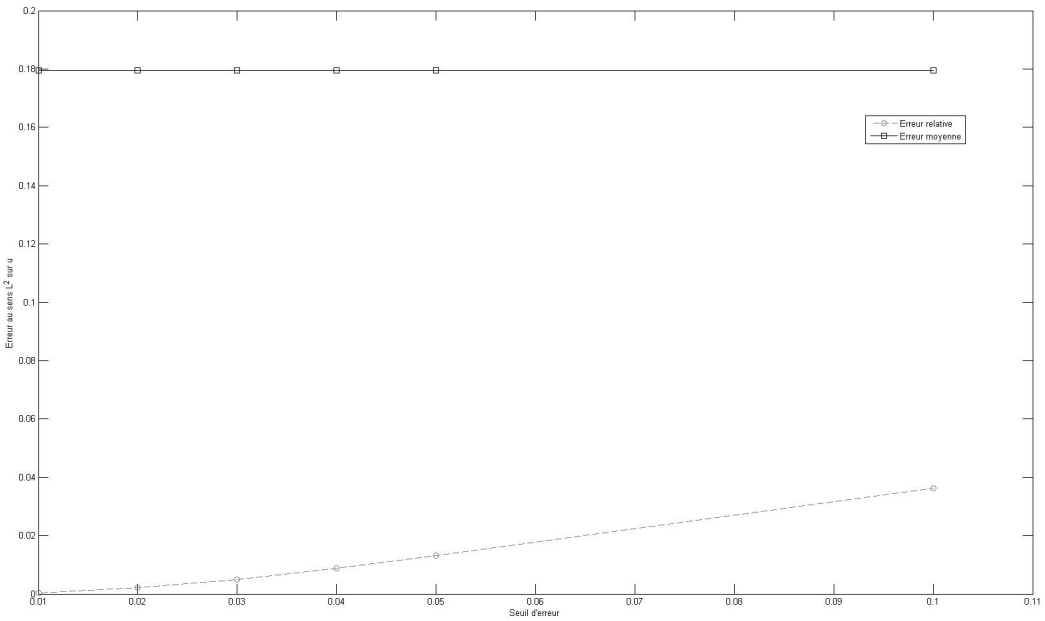

Figure 13: Bogie - Graphe de convergence en moyenne de l'erreur relative en fonction de l'erreur pour $C_v = 0.05$.

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