An Evaluation of Simplified Methods to Compute the Mechanical Steady State

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Abstract The accuracy of the fatigue prediction in low cycle fatigue (LCF) relies on the quality of the crack initiation model, but also on its input. It is then crucial to have a good method to evaluate the mechanical steady state of stresses and strains of the structure. Due to the increasing power of the computers, one can now in some cases run the full computation, for instance with several hundreds of cycles, until a periodic state is reached. This solution may be not adequate when the structure is too big, or when one needs a very quick computation, especially for parametric studies. The purpose of the paper is then to study several techniques providing a direct access to the numerical steady state, and to give a critical review of their performances. Three types of methods will be considered: Zarka's method; cycles skip and the direct cyclic method.

1 INTRODUCTION

Numeric simulation has become essential in mechanical industries to validate structures at the design level. It is often difficult to find a good compromise between the simulation time and its accuracy. Indeed, high reliability in fatigue prediction requires non-linear behaviour rules, a fine meshing that means a high number of degrees of freedom (dof)… Moreover, real structures are often subjected to complex loading. That is why computations to validate components are often very time consuming. To reduce significantly the computation time, simplified methods to reach the steady state of the structure were developed among the last decades. In this paper three methods will be studied: Zarka’s method, cycles skip, the direct cyclic method.

2 ZARKA’S METHOD

This method makes it possible to approach quickly the stabilized macroscopic elastoplastic state of a structure from the history of its elastic state [1,2]. Zarka and Casier [2] define the transformed internal parameter $\underline{y} = \underline{\underline{C}}\underline{\varepsilon_p}$, which can be assimilated to the kinematic hardening. Then, the authors define the structural transformed parameter $\underline{Y} = \underline{y} - dev(\underline{R})$. An example could be useful to understand the principles of this method. For a plane-stress state, Figure 1 represents the $\underline{Y}$ parameter evolution in the $\underline{Y}$ -space (that is a plan here), in the case of elastic and plastic shakedown. The initial yield surface is represented by a dashed convex, yield surfaces at maximum and minimum loading moments by solid convexes, $\underline{S_{el}}$ -path by dashed arrows and $\underline{Y}$ -path by bold solid arrows. In both cases, it appears that $\underline{Y}$ “follows” the convex centered on $\underline{S_{el}}$. In the case of elastic shakedown, the steady-state is reached when the structural transformed parameter is in the common part of the two “min” and “max” convexes, i.e. “plastically admissible”. It should be noted that, following the initial state ($\underline{S_0^{el}}$, $\underline{Y_0}$), the steady state is reached as of the first semi-cycle ($\underline{Y_L^1}$) or after an infinite number of cycles ($\underline{Y_L^2}$).

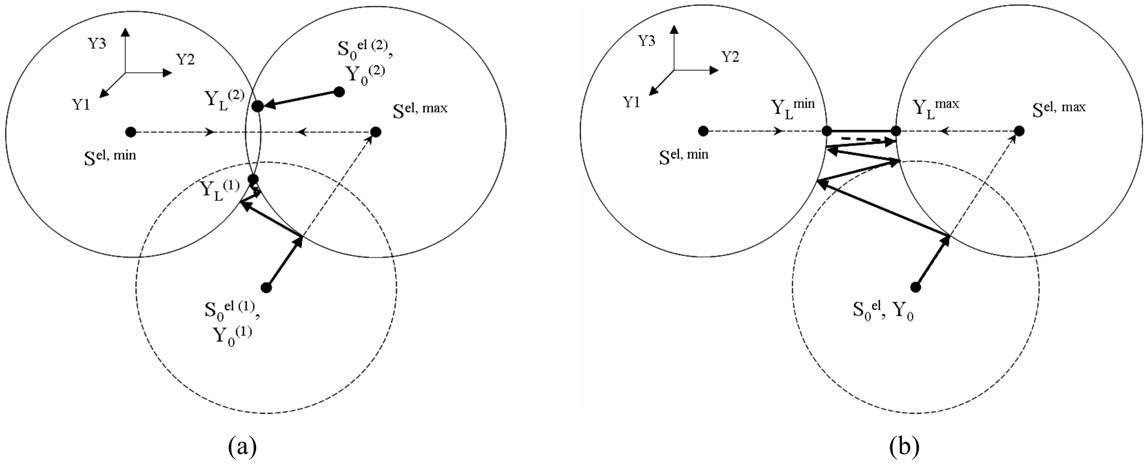


Figure 1 : Y evolution in the case of elastic (a) and plastic (b) shakedown

Once the asymptotic value(s) of $\underline{Y}$ is known, $\underline{R}$ can be determined and then $\underline{\varepsilon}^p$.

3 CYCLES SKIP METHOD

This method [3,4,5] was implemented [6] in the FEM code ZeBuLoN at the Centre des Materiaux of the Ecole des Mines de Paris. It consists in "skipping" cycles by successive extrapolations, and in expressing internal variables in terms of number of cycles. Y is defined as a vector which components are the internal variables expressed as functions of the number of cycles. It can be expressed by a second order Taylor development (equation (1)).

$$\underline{Y}(N+\Delta N)=\underline{Y}(N)+\underline{Y}'(N)\Delta N+\underline{Y}''(N)\frac{\Delta N^2}{2} \tag{1}$$

Supposing that the second order term of the Taylor development is negligible toward the first one, we can introduce a precision factor η such as:

$$\Delta N_i = 2\eta\frac{\underline{Y'_i}(N)}{\underline{Y''_i}(N)} \tag{2}$$

We obtain as many ΔN_i as the total number of components contained in the internal variables; one should choose the smaller one. A value of η = 0,1 gives acceptable results.

4 DIRECT CYCLIC METHOD

The Direct Cyclic Algorithm (DCA) [7,8,9] is an iterative method composed of two steps performed on the whole cycle of the loading history. In the "global step", equilibrium equations are solved. Then in the "local step", constitutive equations are solved. Then we return to the global step, and so on until the global step solution verifies the yield criterion.

4.1 The global step

Knowing the plastic strain, this step consists in finding statically and cinematically admissible solutions. From the equilibrium equation we can deduce the virtual work principle and then an equation that can be expressed in a simplified form (equation (3)).

$$K.U_n(t)=F(t)+Q^p(t) \tag{3}$$

Where K is the stiffness matrix, $U_n(t)$ the displacements at each node of the structure, $F(t)$ the nodal forces resulting from volume and surface external forces, $Q^p(t)$ the nodal forces resulting from the plastic strain. Note that the periodicity of the solution is imposed. That means that at the beginning of the global step, the values of plastic strain and internal variables are replaced by those determined by the local step. This equation can be solved for each increment by the incremental method or in the global step of the direct cyclic method that we describe here. Whereas the incremental method considers the whole loading history, only one cycle is discretized here in N increments and then decomposed in Fourier series, to be expressed in the frequential field.

From the displacement Fourier's coefficients one deduces the strain coefficients. Then, through a Fourier's recomposition we get the total strain and statically admissible stress fields among the whole cycle.

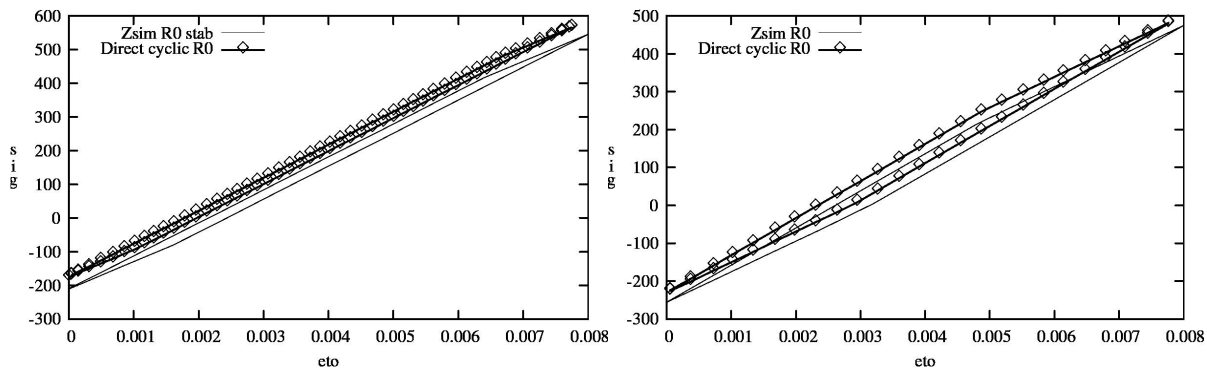
4.2 The local step

For k=1 to N, each total strain increment $\Delta\varepsilon$ between t_k and t_{k+1}, and the plastic strain ε_k^p at t_k are given by the global step. We can now determine the value of the plastic strain and the plastically admissible stress at t_{k+1} [10]. As it is said in 4.1, this value will replace the former plastic strain at the beginning of the global step.

5 DIRECT CYCLIC METHOD EVALUATION

This method was studied on a lump to avoid effects of the meshing. This lump was subjected to a uniaxial tensile stress state. The material considered has a cyclic softening behavior, modelled by coefficients Q and b of the Chaboche model. Direct Cyclic Analysis (DCA) solutions are compared with FEM incremental computations.

In the Figure 2 are presented the steady-state hysteresis in case of repeated (R_ε = 0) strain-controlled loading. In Figure 2(a), the behaviour presents a mean-stress relaxation, not in Figure 2(b).

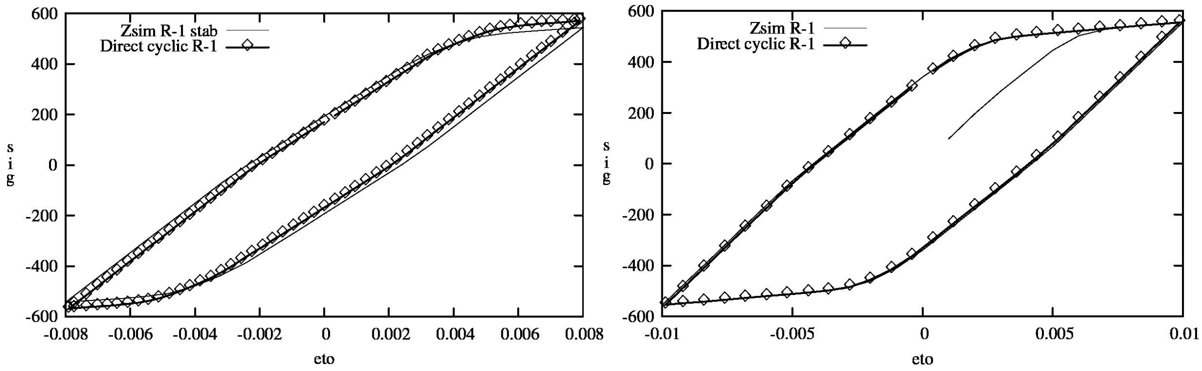


(a) (b)

Figure 2 : Incremental/DCA comparison; (a) with mean-stress relaxation (b) without mean-stress relaxation

It can be observed that for R_ε = 0, the incremental and DCA solutions do not have the same mean stress, whether the cyclic softening behaviour (a) or the cyclic stabilized one (b) is used. It can be deduced that the shift observed in Figure 2(a) is not due to the mean-stress relaxation but only to a non-zero mean stress; indeed, the same shift is observed in both cases.

In Figure 3 are plotted the σ-ε steady loading pathes for R_ε = -1. As there is no mean stress, no vertical shift is observed. However, when using the cyclic softening behaviour (Figure 3 (a)), we observe an error on the stress amplitude. This error does not appear when using the cyclic stabilized behaviour (b), thus it seems to be due to the fact that DCA cannot take into account isotropic hardening.

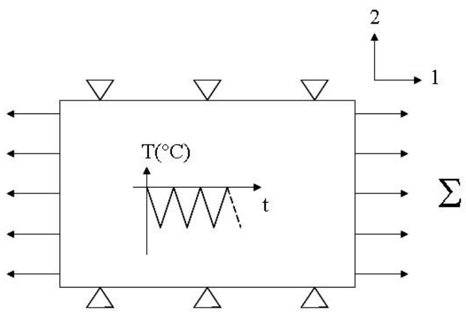


(a) (b)

Figure 3 : Incremental/DCA comparison (a) with cyclic softening behaviour, (b) with cyclic stabilized behaviour

6 CYCLES SKIP AND ZARKA'S METHODS EVALUATION

A simple anisothermal loading case is hereafter illustrated. A 2D plane-stress lump is subjected a static tensile stress $\Sigma = \sigma_{11} = 180$ MPa in direction 1, a null total deflection in direction 2, and a cycling temperature from 0 to -3 °C. Three loading cases are considered here: asymptotic elastic shakedown, rapid plastic shakedown and slow plastic shakedown. The materials properties and loading geometry are described in Figure 4:

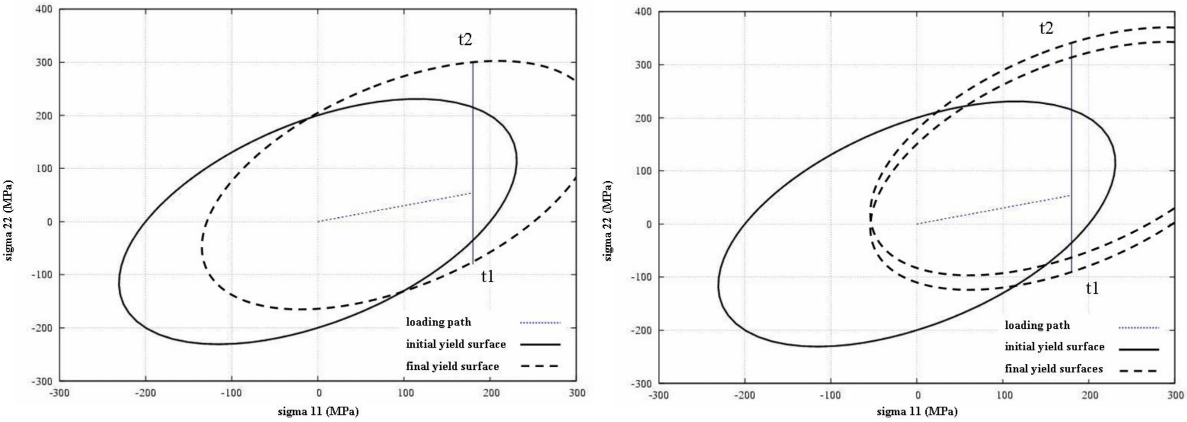


E (Gpa)	200
ν	0,3
σ_y (Mpa)	200
α (°C^{-1})	0,001

(a) (b)

Figure 4 : loading geometry (a) and material properties (b)

Figure 5 represents the loading path in plane $\sigma_{22} - \sigma_{11}$, the initial yield surface position, and the yield surface position at the extreme moments of the loading history when the steady-state is reached. In case (a) (elastic shakedown), the yield surface "follows" the loading path during alternated plastifications until elastic shakedown is reached, i.e. until the yield surface totally includes the stabilized loading path t1-t2. In case (b), a plastic shakedown occurs, i.e. the yield surface cannot totally include the stabilized loading path t1-t2. Plastification occurs at each half-cycle: the yield surface moves alternatively to points t1 and t2.

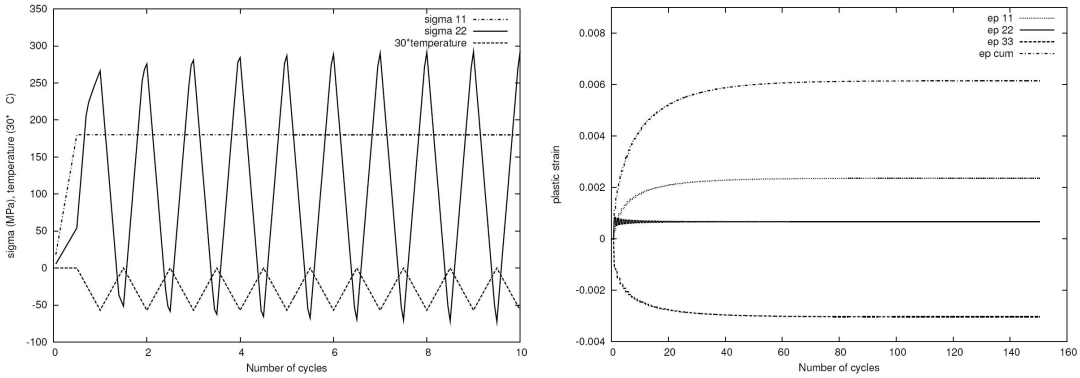


(a) (b)

Figure 5 : loading path and yield surface displacement for (a) elastic shakedown and (b) plastic shakedown

6.1 Elastic shakedown

In this case, the static tensile stress is equal to 180 Mpa, the temperature varies from 0 to –1,9°C. The exact steady state would be reached after an infinite number of cycles. An elastoplastic FEM simulation with ZeBuLoN on a lump was performed; Figure 6 represents the evolution of several variables among cycles until the steady state is reached.



(a) (b)

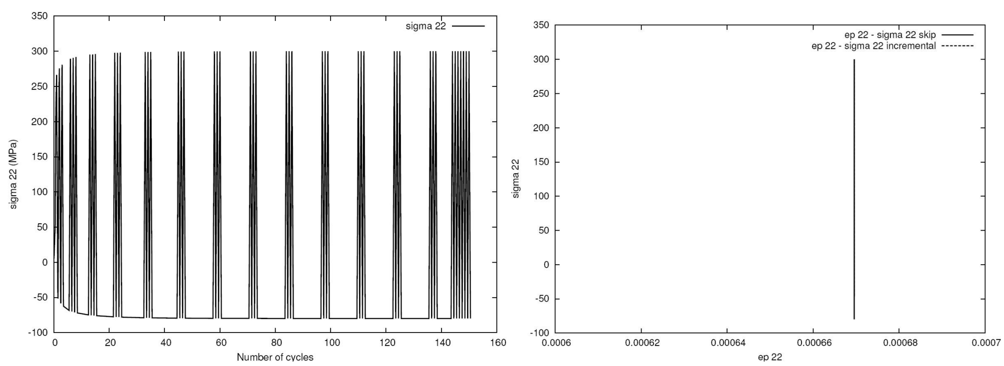
Figure 6 : FEM solution in case of elastic shakedown :
(a) Evolution of σ_{11}, σ_{22}, and temperature (b) Evolution of plastic strains ε^p_{11}, ε^p_{22}, ε^p_{33} and cumulated ε^p

The solution was also computed by Zarka's method with code ZAC distributed by firm CADLM, some results are taken from [11]. It was also computed by cycles skip method, implemented in code ZeBuLoN of the Centre des Matériaux. FEM, analytical, Zarka's method and cycles skip method solutions for σ_{22}, ε^p_{11} and ε^p_{22} are presented in Table 1, where t1 and t2 are the times of the loadings extrema. Computation (CPU) time is also presented. Zarka's and cycles skip method give good results, but the first method is faster than the second one. Moreover, most of the computation time may seems to be due to the ZAC software organization, not to the method itself. However, one should note that only linear kinematic hardening can be used with Zarka's method, whereas cycles skip one makes it possible to use non-linear kinematic hardening.

Table 1 : Zarka's and cycle skip method comparison for elastic shakedown

	FEM	Analytical	Zarka	Cycles skip
ε^p_{11} (t1, t2)	$2{,}3624.10^{-3}$	$2{,}3624.10^{-3}$	$2{,}3623.10^{-3}$	$2{,}3624.10^{-3}$
ε^p_{22} (t1, t2)	$6{,}6957.10^{-4}$	$6{,}6957.10^{-4}$	$6{,}6964.10^{-4}$	$6{,}6957.10^{-4}$
σ_{22} (t1) / MPa	-79,91	-79,91	-79,91	-79,91
σ_{22} (t2) / MPa	300,09	300,09	300,09	300,09
CPU time (s)	196,6	/	28,0	65,4

Figure 7 illustrates the cycles computed by cycles skip method during the whole loading history (a) and the steady states reached by a complete incremental computation and cycle skip method (b), that are similar. The less non-linear the behaviour is, the more cycles can be skipped.



(a) (b)

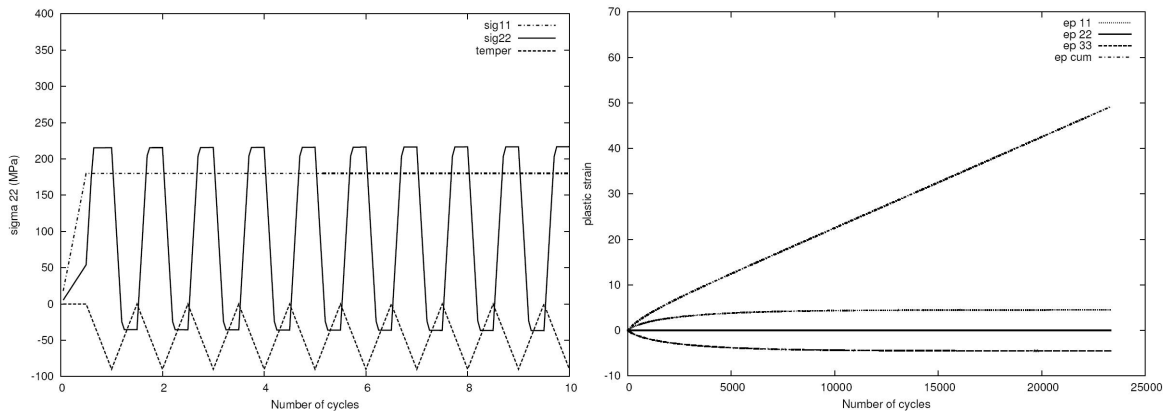
Figure 7 : (a) cycles computed with cycles skip method; (b) incremental and cycles skip steady states

One should note that the η factor can be adjusted following the expected precision. Here, a factor of 0,1 is used and makes it possible to reach a quite good approximation of the steady-state. But a 0,5 factor gives an acceptable solution (σ_{22} (t2) = 300,06 MPa), avoiding the computation of many cycles.

6.2 Plastic shakedown

6.2.1 Case of a small kinematic hardening

In this case, the static tensile stress is equal to 180 Mpa, and the temperature varies also from 0 to 3,0°C. The kinematic hardening coefficient is H = 30 MPa; one needs a large number of cycles to reach the steady state. Figure 8 represents the evolution of several variables among the cycles.



(a) (b)

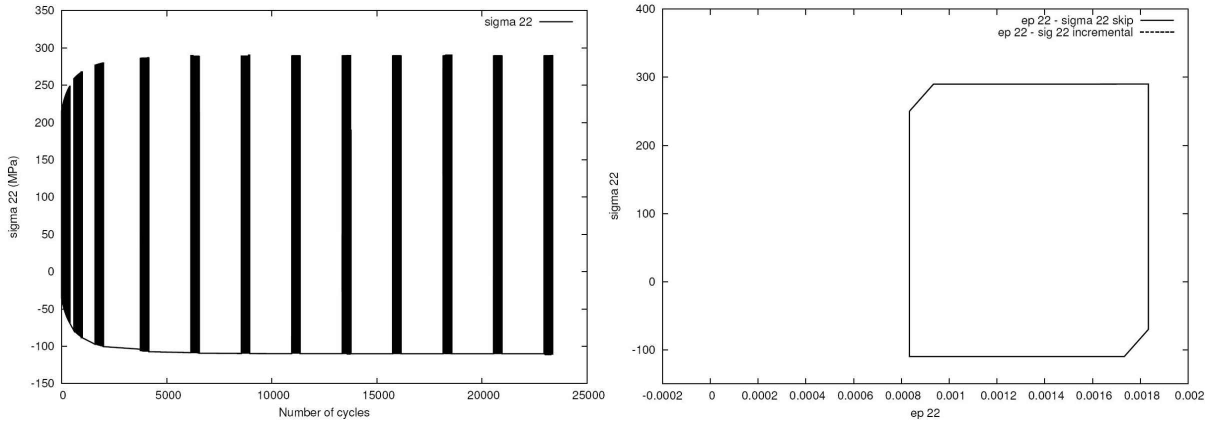
Figure 8 : FEM solution in case of plastic shakedown, H = 30 MPa
(a) Evolution of σ_{11}, σ_{22}, and temperature (b) Evolution of plastic strains ε^p_{11}, ε^p_{22}, ε^p_{33} and cumulated ε^p

FEM, analytical, Zarka's method and cycles skip method solutions for σ_{22}, ε^p_{11} and ε^p_{22} are presented in Table 2. Values of ε^p_{22} given by Zarka's method are said to be erroneous because of the lack of precision of CASTEM elastic computation [11]. Because of the very low kinematic hardening coefficient, the steady state could not be completely reached by FEM computation after 23300 cycles. This "pseudo-steady" state has to be compared with cycle skip one, that was also obtained after 23300 cycles. Zarka's results should be compared to the analytical ones. One should pay attention to values of CPU time: for a large kinematic hardening, Zarka's method is by far the fastest method.

Table 2: Zarka's and cycle skip method comparison for plastic shakedown (H = 30 MPa)

	FEM		Analytical		Zarka		Cycles skip	
	t1	t2	t1	t2	t1	t2	t1	t2
ε^p_{11}	4,4968	4,4963	4,4989	4,4984	4,4990	4,4985	4,4964	4,4959
ε^p_{22}	$8,1988.10^{-4}$	$1,8197.10^{-3}$	$8,1988.10^{-4}$	$1,8197.10^{-3}$	erroneous	erroneous	$6,9133.10^{-4}$	$1,6915.10^{-3}$
σ_{22} / MPa	-109,98	290,05	-109,98	290,05	-109,98	290,05	-109,98	290,05
CPU time (s)	31192,9		/		53,2		3309,1	

Figure 9 illustrates the cycles computed by cycles skip method during the whole loading history (a) and the steady states reached by a complete incremental computation and cycle skip method (b).

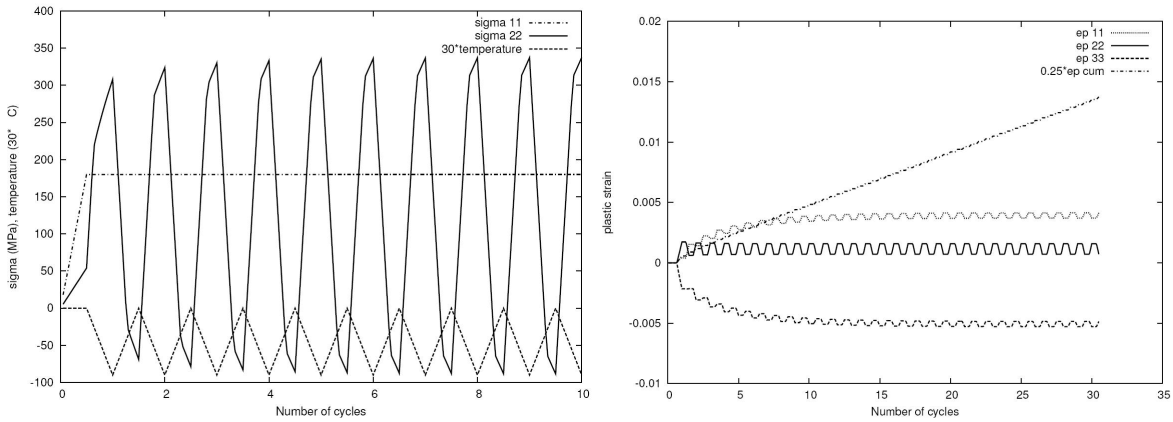


(a) (b)

Figure 9 : (a) cycles computed with cycles skip method; (b) incremental and cycles skip steady states

6.2.2 *Case of a large kinematic hardening*

In this case, the static tensile stress is equal to 180 Mpa, and the temperature varies also from 0 to 3,0°C. But the kinematic hardening coefficient is H = 30000 MPa, so that the steady state is almost reached after a few cycles. Figure 10 represents the evolution of several variables among cycles.



(a) (b)

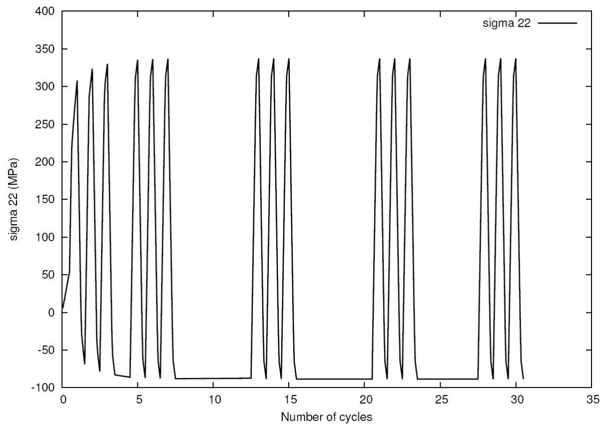
Figure 10 : FEM solution in case of plastic shakedown, H = 30 000 MPa
(a) Evolution of σ_{11}, σ_{22}, and temperature (b) Evolution of plastic strains ε^p_{11}, ε^p_{22}, ε^p_{33} and cumulated ε^p

FEM, analytical, Zarka's method and cycles skip method solutions for σ_{22}, ε^p_{11} and ε^p_{22} are presented in Table 3. One should note that for a large kinematic hardening, Zarka's method presents no obvious advantage when compared to the cycles skip one, especially regarding the CPU time.

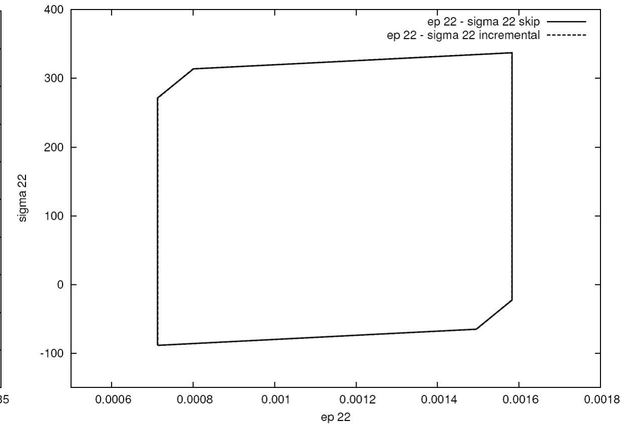
Table 3 : Zarka's and cycle skip method comparison for plastic shakedown (H = 30 000 MPa)

	FEM		Analytical		Zarka		Cycles skip	
	t1	t2	t1	t2	t1	t2	t1	t2
ε^p_{11}	$4{,}1435.10^{-3}$	$3.7087.10^{-3}$	$4{,}1435.10^{-3}$	$3.7087.10^{-3}$	$4{,}1435.10^{-3}$	$3{,}7087.10^{-3}$	$4{,}1410.10^{-3}$	$3{,}7058.10^{-3}$
ε^p_{22}	$7{,}1304.10^{-4}$	$1{,}5826.10^{-3}$	$7{,}1304.10^{-4}$	$1{,}5826.10^{-3}$	$7{,}1300.10^{-4}$	$1{,}5827.10^{-3}$	$7{,}1304.10^{-4}$	$1{,}5826.10^{-3}$
σ_{22} / MPa	-88,61	337,48	-88,61	337,48	-88,61	337,48	-88,61	337,48
CPU time (s)	56,2		/		27,6		9,9	

Figure 11 illustrates the cycles computed by cycles skip method during the whole loading history, and a comparison between the steady state computed incrementally and by cycle skip method.



(a)



(b)

Figure 11 : (a) cycles computed with cycles skip method; (b) incremental and cycles skip steady states

7 DISCUSSION

The purpose of this paper was to evaluate three simplified methods used to reach the steady state: Zarka's, direct cyclic, and cycles skip methods. From the calculations performed it would not be appropriated to compare one method to each other because they have not the same features and thus, one has to use them according to the application. For example, even if Zarka's method is faster than the cycles skip one, it cannot be used with a non-linear kinematic hardening. In this case, only small strain amplitude cycles give rather good results, because the slope at the working point of the hardening curve has to be extrapolated. However, for a small kinematic hardening, cycle skip method is more time consuming, so that using a linear kinematic hardening may be a good compromise to reach a first approximation. Regarding the direct cycle method, one could say that it is faster than cycles skip one. Indeed, it needs no incremental computation; however, a shift is observed in stresses when strain is imposed. That means that this method should not be used with stress or strain criteria. No error is observed on the energy; that is why this method is particularly recommended with use of energy fatigue criteria, based on critical plane or on dissipation.

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